UNVERSAL FEATURES OF THE ORDER-PARAMETER FLUCTUATIONS

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We discuss the universal scaling laws of order parameter fluctuations in any system in which the second-order critical behavior can be identified. These scaling laws can be derived rigorously for equilibrium systems when combined with the finite-size scaling analysis. The relation between order parameter, criticality and scaling law of fluctuations has been established and the connexion between the scaling function and the critical exponents has been found.

1 Introduction

Averaged quantities are still commonly used for the description of many complex processes in physics, chemistry or astrophysics. This kind of approach is inprinted in our minds by an unquestionable success of thermodynamics in which fluctuations around the thermodynamic value are small in large systems. It is our experience that the gross measures in, *e.g.*, the particle reaction data are not sufficient to discriminate among models unless supplemented with more fine grained information, especially fluctuations and correlations of various kinds. Recent resurgence of interest in fluctuations in the strong interaction physics is due to the experimental possibility to measure event by event many strongly fluctuating quantities, *e.g.*, the multiplicity of produced particles in ultrarelativistic collisions of leptons, hadrons, nuclei, or the charges (masses) of fragmentation products of highly excited heavy-ion residue. The immediate question is then what is the information contained in fluctuations and, in particular, in the order parameter fluctuations , how do these fluctuations scale with the system-size and what is the relation between criticality and the probability distribution of order parameter fluctuations.

2 Order parameter fluctuations

Let us suppose that the thermodynamic free energy of finite system depends on three parameters : η (the intensive order parameter), ϵ (the distance to the critical point) and N (the size of the system). Widom[1] has proposed that close to the critical point, the free energy density in the thermodynamic limit scales as :

$$f_o(\lambda^\beta \eta, \lambda \epsilon) \sim \lambda^{2-\alpha} f_o(\eta, \epsilon) \quad , \tag{1}$$

where α, β are usual critical exponents. There is no critical behavior in a finite system. However, a finite system behaves like an infinite one if the correlation length ξ becomes comparable to the typical length L of the system. This is basically the argument of Fisher and Barber[2] leading to the finite-size scaling analysis of critical systems. The pseudocritical point for a finite system appears at a distance $\epsilon \sim cN^{-1/\nu d}$ from the critical point[3] , where c is some dimensionless constant which can be either positive or negative[4]. One can then deduce the scaling of critical free energy density at this point :

$$f_c(\eta, N) \sim \eta^{\frac{2-\alpha}{\beta}} \phi(\eta N^{\frac{\beta}{\nu d}}) \quad . \tag{2}$$

Assuming the hyperscaling relation : $2 - \alpha = \nu d$, and using Rushbrooke relation between critical exponents : $\alpha + 2\beta + \gamma = 2$, one can write the total free energy $F(\eta, \epsilon, N) = N f_o(\eta, \epsilon)$ at the pseudocritical point as follows :

$$F_c(\eta, N) \sim f_o(\eta N^{\frac{\beta}{\gamma+2\beta}}, c) \quad . \tag{3}$$

The canonical probability density $P[\eta]$ to get some value of the order parameter η is given by[5] :

$$P[\eta] = Z_N^{-1} \exp(-\beta_T F(\eta, \epsilon, N)) \quad , \tag{4}$$

where β_T is the inverse of temperature. Using Eq. (3), one obtains the partition function :

$$Z_N \sim N^{-\frac{\beta}{\gamma+2\beta}} \sim <|\eta|> \quad . \tag{5}$$

It is then easy to see that the probability density $P[\eta]$ obeys the first scaling law :

$$<|\eta|> P[\eta] = \Phi(z_{(1)}) = \Phi\left(\frac{\eta - <|\eta|>}{<|\eta|>}\right)$$
$$= a(\beta_T) \exp\left(-\beta_T f_o\left(\frac{\eta}{<|\eta|>}, c\right)\right) \quad , \tag{6}$$

with the constant coefficient β_T independent of η , and the scaling function $\Phi(z_{(1)})$ depending on a single scaled variable : $z_{(1)} = (\eta - <|\eta|>)/<|\eta|>$. The scaling limit is defined by the asymptotic behaviour of $P[\eta]$ when $\eta \to \infty$, $<|\eta|> \to \infty$, but $(\eta/<|\eta|>)$ has a finite value. The temperature-dependent factor $a(\beta_T)$ is determined by the normalization of $P[\eta]$. One may notice that the logarithm of scaling function (6) corresponds to the non-critical free energy density at the renormalized distance $\epsilon = c$ from the critical point. If the order parameter corresponds to the cluster multiplicity, like in the fragmentation - inactivation binary (FIB) process[6,7], then (6) can be written in an equivalent form to the KNO scaling[8] , proposed some time ago as the ultimate symmetry of S - matrix in the relativistic field theory[9] . Relation between the KNO scaling and the phase transition in Feynman-Wilson gas as well as the criticality of self-similar FIB process was studied as well[10,11] . Defining the anomalous dimension for an extensive quantity $N\eta$ as :

$$g = \lim_{N\to\infty} g_N = \lim_{N\to\infty} \frac{d}{d\ln N} \left(\ln <N|\eta|> \right) \quad , \qquad (7)$$

one can see that due to (5), the scaling (6) holds when $g = (\gamma + \beta)/(\gamma + 2\beta)$. Consequently, g is contained between $1/2$ and 1. Whenever the cluster-size can be reasonably defined for the second-order transition, like it is the case in percolation, Ising model or Fisher droplet model, the exponent τ of the power-law cluster-size distribution : $n(k) \sim k^{-\tau}$, satisfies additional relations[12] : $\gamma + \beta = 1/\sigma$ and $\gamma + 2\beta = (\tau - 1)/\sigma$, which yield : $g \equiv 1/(\tau - 1)$. This means that τ has to be contained between 2 and 3 when (6) holds.

What may happen if the order parameter is not known exactly? To illustrate this point, let us consider a quantity : $m = N^\kappa - N\eta$, where η is the true order parameter and κ is larger than the anomalous dimension g . For large N , $|m|$ is of order N^κ . Writing (6) with m instead of η, and taking into account : $P[\eta]d\eta = P[m]dm$, one finds the delta - scaling :

$$<|m|>^\delta P[m] = \Phi(z_{(\delta)}) \equiv \Phi\left(\frac{m - <|m|>}{<|m|>^\delta}\right) \quad , \qquad \delta = \frac{g}{\kappa} < 1 \quad , \quad (8)$$

with the scaling function $\Phi(z_{(\delta)})$ depending only on the scaled variable : $z_{(\delta)} = (m - <|m|>)/<|m|>^\delta$. According to (3) and (6), the logarithm of scaling function :

$$\ln \Phi(z_{(\delta)}) = -\beta_T f_o(z_{(\delta)}, c) \quad , \qquad (9)$$

is directly related to the non-critical free energy , in either ordered $(c > 0)$ or disordered $(c < 0)$ phase. The relation $\delta = g/\kappa$ in (8) singularizes importance of an extensive variable : $m = N(1 - \eta) \equiv N\hat\eta$. $\hat\eta$ can be useful in

3

phenomenological applications and plays an important role in the percolation studies[12] . One finds for this choice : $< m >\sim N$, with algebraic finite-size corrections, and the delta-scaling (8) with $\delta = g$. Hence, $P[N\hat{\eta}]$ allows to determine the anomalous dimension g and, consequently, the ratio of critical exponents β and γ .

Let us suppose now that the extensive parameter m is not critical , *i.e.*, either the system is in a critical state but the parameter m is not critical, or the system is outside of critical region. The value of m at the equilibrium is obtained by minimizing the free energy. Let us suppose that the free energy F is analytical in the variable m close to its most probable value m^*, *i.e.*, :

$$F \sim N^{-\phi}(m - m^*)^{\phi+1} \quad . \tag{10}$$

In most cases $\phi = 1$, though, in general, ϕ can take any positive integer value. Using (10) one obtains : $< |m| >\sim \mu^* N$, where μ^* is a positive (finite) number independent of N , and :

$$Z_N \sim N^{\frac{\phi}{\phi+1}} \sim \quad < |m| >^{\frac{\phi}{\phi+1}} \quad . \tag{11}$$

The probability density $P[m]$ verifies the generalized scaling law (8) :

$$< |m| >^\delta P[m] = \Phi(z_{(\delta)}) = \exp\left(-\beta_T \mu^{*\phi}\left(\frac{(m- < |m| >)}{< |m| >^{\frac{\phi}{\phi+1}}}\right)^{\phi+1}\right) \quad ,\tag{12}$$

but now $\delta \ (= \phi/(\phi + 1) < 1)$ is constrained by the value of ϕ . In the generic case $\phi = 1$, δ equals 1/2 and the scaling function is Gaussian[14]. The second scaling (12) holds for $< m >\sim N$ but now with the exponential finite-size corrections.

3 Results

The above results apply to any *second* order transition, and , in particular, they are not limited to the Landau-Ginzburg theory of phase transitions. As an illustration, let us look at the bond percolation model (Fig. 1). The intensive order parameter is the normalized mass of largest cluster : $\eta = S_{max}/N$. Fig. 1a (the upper left plot) shows the scaling function $\Phi(z_{(1)})$ in $3D$-percolation at the value of bond activation probability ($p_{cr} = 0.2482$) corresponding to the critical activation in the infinite system. Results for different lattice sizes are superposing well in the scaled variables (8) for $\delta = 1$. One should recall that $\Phi(z_{(1)})$ is also a 'portray' of the free energy density at a renormalized distance from the critical point. The probability distribution $P[N\eta]$ changes
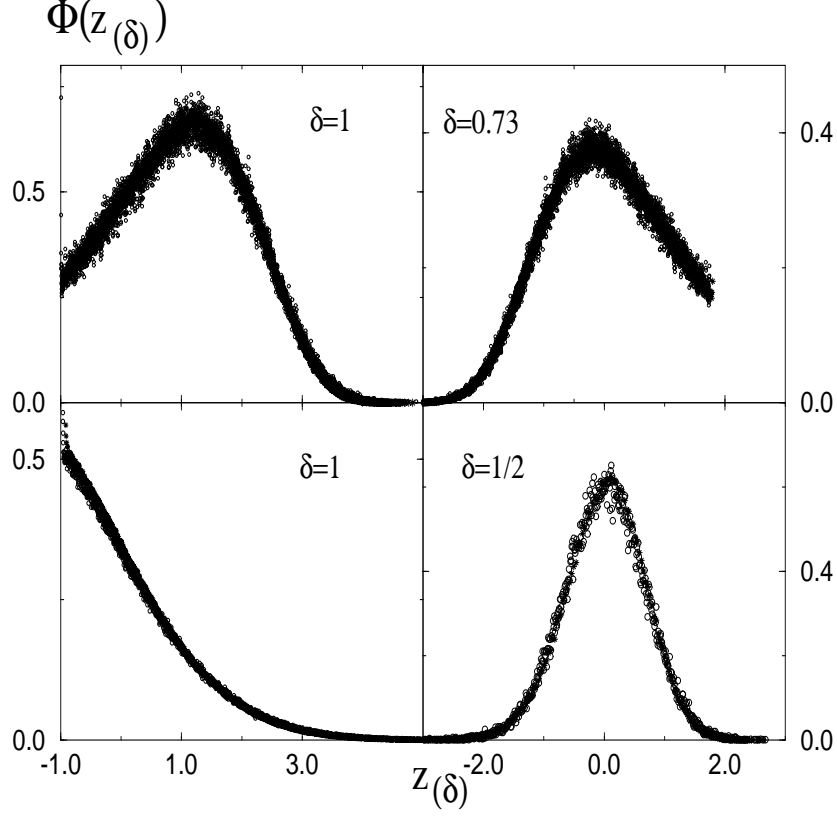
4

Figure 1: The order parameter fluctuations in the bond percolation model in 3D are calculated in the lattices : $N = 10^3$ and $N = 14^3$, and plotted in the scaling variables of delta-scaling (8). **(a)** (the upper left plot) : for the bond activation probability : $p_{cr} = 0.2482$; **(b)** (the upper right plot) : for $p = 0.245 \simeq p_{cr}$; **(c)** (the lower left plot) : for $p = 0.35 > p_{cr}$; **(d)** (the lower right plot) : fluctuations of the quantity : $m = M_1 - S_{max}$, where $M_1 = \sum_k kn(k)$ is the first moment of the fragment-size probability distribution, at the percolation threshold p_{cr} are plotted in the scaling variables of delta-scaling (8) with $\delta \simeq 0.80$. For more details, see the description in the text.

rapidly, as can be seen in Figs. 1a and 1b (the upper right plot). The latter plot is obtained for the bond activation probablity $p = 0.245$, very close to p_{cr}. Results in Fig. 1b are plotted in the scaling variables (8) for $\delta = 1$, even though slight deviations between different calculations can be seen in the shoulder region and in the tail for positive $z_{(1)}$. Fig. 1c (the lower left plot) shows the order parameter distribution in the 'liquid phase' $(p = 0.35 > p_{cr})$. One finds the second scaling (12) , in agreement with the analytical derivation. Finally, Fig. 1d (the lower right plot) shows what happens at the percolation threshold $(p = p_{cr})$ if instead of $P[N\eta]$, one plots the probability distribution of $m = M_1 - S_{max}$, where $M_1 = \sum_k k n(k)$ is the first moment of the fragment-size probability distribution. m $(\equiv N\hat{\eta}$), is related in a non-trivial way to the order parameter and, in particular, it conserves the singularity of S_{max} . $P[m]$ obeys the delta-scaling (8) for $\delta \simeq 0.80$, what should be compared with : $1/(\tau - 1) = 0.84$, given by the analytical argument leading to the delta-scaling (8) . This signature of phase transition disappears for $p \neq p_{cr}$, i.e., δ becomes equal $1/2$.

As a second example, let us consider the FIB model [6,7] which exhibits the second order, shattering phase transition [13]. In this off-equilibrium case, analytical derivation of (6) and (8), in principle, does not apply. One deals in FIB model with clusters characterized by a conserved quantity, called the cluster mass. The anscestor cluster of mass N is fragmenting via an ordered and irreversible sequence of steps until either the cutoff-scale for monomers is reached or all clusters are inactive.

Each step in this cascade is either a binary fragmentation of an active cluster $(k) \rightarrow (j) + (k - j)$ with a fragmentation rate $\sim F_{j,k-j}$ (a mass partition probability), or its inactivation $(k) \rightarrow (k)^*$ with an inactivation rate $\sim I_k$. The order parameter is here the reduced cluster multiplicity M_0/N or the reduced monomer multiplicity, both of them closely interrelated. The total fragmentation probability p_F at each step of FIB cascade is :

$$p_F(k) = \sum_{j=1}^{k-1} F_{j,k-j} \ (I_k + \sum_{j=1}^{k-1} F_{j,k-j})^{-1} \quad . \tag{13}$$

The cluster mass independence of $p_F(k)$ at any step until the cutoff-scale characterizes the *critical transition region*. FIB process is self-similar in this regime. For $p_F > 1/2$, the anomalous dimension (7) (now $N|\eta| \equiv M_0$) varies from 0 to 1, what is different from the limits on g in the equilibrium systems. The average multiplicity of inacitve clusters is [11] : $< M_0 > \sim N^{\tau-1}$ $(1 < \tau < 2)$, and $g = \tau - 1$. In the *shattered phase*, the average multiplicity is : $< M_0 > \sim N$, the cluster-size distribution is a power-law with $\tau > 2$ and
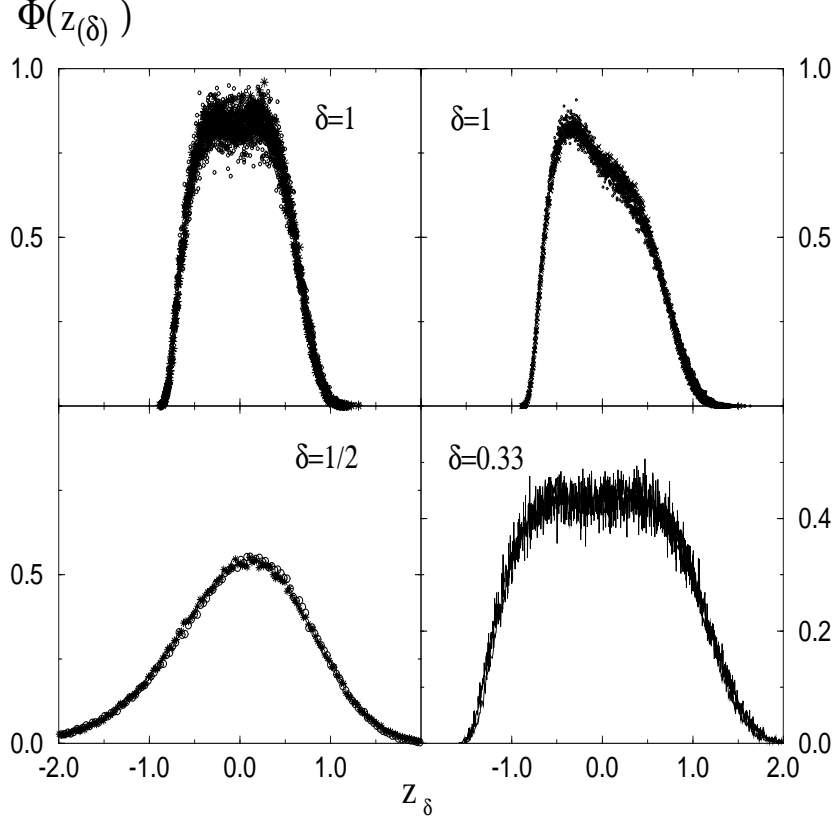
Figure 2: The order parameter fluctuations in the FIB model calculated for two initial sizes : $N = 2^{10}$ and $N = 2^{14}$, and plotted in the scaling variables of delta-scaling (8). **(a)** (the upper left plot) : the critical FIB process for $p_F = 0.875$, $a = 0$ corresponding to the anomalous dimension $g = 0.75$; **(b)** (the upper right plot) : fluctuations of the quantity : $m = N - M_0$ are plotted in the scaling variables of delta-scaling (8) with $\delta \simeq 0.73$, **(c)** (the lower left plot) : the critical FIB process for $p_F = 0.7$, $a = 0$ corresponding to the anomalous dimension $g = 0.4$; **(d)** (the lower right plot) : shattered phase for $a = 0,\ b = -1,\ \tau = 4$. For more details, see the description in the text.

the anomalous dimension is $g = 1$. In this phase, p_F is an increasing function of cluster mass k and, hence, the FIB cascade is not self-similar. Most of the interesting physical applications correspond to homogeneous fragmentation functions [7] : $F_{\lambda j, \lambda(N-j)} = \lambda^{2a} F_{j, N-j}$, e.g., $F_{j, N-j} \sim [j(N-j)]^a$. For the homogeneous inactivation rate-function : $I_k \sim k^b$, the critical transition region in FIB model corresponds to the line : $b = 2a + 1$, and the shattered phase corresponds to : $b < 2a + 1$. Such homogeneous rates : $F_{j, k-j}$ and I_k , will be used in examples shown in Fig. 2. Fig. 2a (the upper left plot) shows the scaling function $\Phi(z_{(1)})$ of critical FIB process for $p_F = 0.875$, $a = 0$, what yields $g = 0.75$. The asymmetry of $\Phi(z_{(1)})$ about $z_{(1)} = 0$, is rather common in the critical sector of FIB model [11]. This sector and its characteristic first scaling (6) extends to the domain $0 < g < 1/2$ [11] , excluded in equilibrium models. In this domain $(g < 1/2)$, the most probable value of $\Phi(z_{(1)})$ is at $z_{(1)} = -1$, whereas for $g > 1/2$ it takes a value close to $z_{(1)} = 0$ [11] . This can also be seen in Fig. 2c (the lower left plot) which exhibits the scaling function $\Phi(z_{(1)})$ of critical FIB process for $p_F = 0.7$, $a = 0$, for which $g = 0.4$. What happens if instead of $P[M_0] \equiv P[N\eta]$, one plots $P[N - M_0] \equiv P[N\hat{\eta}]$, as shown in Fig. 2b (the upper right plot). Analogously as in percolation (see Fig. 1b), $P[N\hat{\eta}]$ is scaling (8) with the non-trivial exponent $\delta \simeq 0.73$ which is close to the value $\delta = g = 0.75$, obtained using analytical arguments (8). The order parameter distribution in the shattered phase $(a = 0, b = -1, \tau = 4)$, is shown in Fig. 2d (the lower right plot) in the variables of second scaling (12) for $\delta = 1/2$. Again, one finds an analogy to the situation in the 'liquid' phase of percolation (Fig. 1c).

4 Outlook

The off-equilibrium FIB model shows essentially the same relation between criticality and scaling of order parameter fluctuations as it has been derived analytically for the second-order equilibrium transitions (Eqs. (6), (8), (12)). This is related to the underlying self-similarity which is common to both equilibrium and off-equilibrium realizations of the second-order transition. In that sense, the scaling laws (Eqs. (6), (8), (12)) are the salient features of *any* system exhibiting the second-order transition and the function $\Phi(z_{(\delta)})$ is a fingerprint of the system and its transition. It is therefore important to study implications of scaling (6) on statistical properties of the system. Let us first postulate the definition of the pseudo-free energy :

$$\mathcal{F} = -\tilde{\beta}_T^{-1} \ln(< |\eta| > P[\eta]) \quad , \tag{14}$$

8

with a coefficient $\tilde{\beta}_T$ which is independent of η and characterizes the homogeneous system. If we suppose that the first scaling (6) holds and employing the asymptotical definition of the anomalous dimension (7), one finds :

$$\mathcal{F}/N \sim \eta^{1/(1-g)}\phi(\eta N^{1-g}) \quad , \tag{15}$$

what formally corresponds to (2) with $(1-g)$ instead of both $\beta/(2-\alpha)$ and $\beta/(\nu\delta)$. $\mathcal{F}$ appears to be a pseudo-free energy for *constrained* η, and *fixed* both N and $\tilde{\beta}_T$. This derivation does not suppose that the system is close to the second-order transition in thermodynamical equilibrium. In particular, (6) and (7) are expected to hold for any critical behaviour : at the thermodynamic equilibrium as we have shown rigorously, but also for the non-thermodynamic equilibrium like in percolation, for the off-equilibrium final state like in FIB model, or for the self-organized criticality [16]. The basic quantity is $\mathcal{F}$, which contains thermodynamically relevant information about the analogue of the inverse of temperature. The scaling law of $\mathcal{F}$ (15) gives in turn information about the critical behavior of the system. Finally, let us remind that the formulae (15) contains only one exponent g. To have an access to other critical exponent, we need to vary the field conjugate to the order parameter.

To summarize, we have found the new approach to study critical phenomena both in equilibrium and off-equilibrium systems. This approach is based on the existence of universal scaling laws in the probability distribution of both the order parameter and its complement, in the second order phase transitions. The precise relation between the scaling functions $\Phi(z_{(\delta)})$, the nature of order parameter and the critical exponents yields a new tool for determining the combinations of critical exponents even in small systems and for learning about the nature of critical phenomenon. We hope, this approach will be useful in many phenomenological applications in the strong interaction physics and in the condensed matter physics.

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